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Existence Results for Partial Neutral Functional Differential Equations with Nonlocal Conditions

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We study a nonlocal Cauchy problem for a class of quasi-linear partial neutral functional differential equations with unbounded delay described in the form $\frac{d}{dt}(x(t)+F(t,x_t)) = Ax(t)+G(t,x_t)$, where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space X and F, G are appropriate continuous functions. October, 2001 ICMC-USP

1. INTRODUCTION

The purpose of this paper is to prove the existence of mild, strong and classical solutions for a partial neutral functional differential equation with nonlocal condition modeled in the form

$$\begin{aligned} \tfrac{d}{dt}(x(t)+F(t,x_t)) &= Ax(t)+G(t,x_t), \qquad t\in[0,T], \\ x_\sigma &= \varphi + q(u_{t_1},u_{t_2},u_{t_3},...,u_{t_n}) \in \Omega, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t))_{t\geq 0}$, on a Banach space X; the histories $x_t : (-\infty,0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $\Omega \subset \mathcal{B}$ is open; $0 \leq \sigma < T$; $\sigma < t_0 < t_1 < .... < t_n \leq T$; and $q : \mathcal{B}^n \to \mathcal{B}$; $F, G : [\sigma, T] \times \Omega \to X$ are appropriate continuous functions.

There exist a extensive literature of differential equations with nonlocal conditions. Motivated by physical applications, Byszewski studied in [1] a nonlocal Cauchy problem modeled in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t)+f(t,x(t)), \qquad t\in(\sigma,T], \\ x(0) &= x_0 + q(t_1,t_2,t_3,...,t_n,u(\cdot)) \in X, \end{aligned} \tag{2}$$

where A is the infinitesimal generator of a C_0 semigroup of linear operators on X; $f : [\sigma,T]\times X \to X$, $q : [\sigma,T]^n \times X \to X$ are appropriated functions and the symbol $q(t_1,t_2,t_3,...,t_n,u(\cdot))$ is used in the sense that " $\cdot$ " can be substitute only for the points t_i, for instance $q(t_1,t_2,t_3,...,t_n,u(\cdot)) = \sum_{i=1}^n \alpha_i u(t_i)$. In the cited paper, Byszewski show the

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existence of mild, strong and classical solutions for the nonlocal problem (2) using usual assumptions on the functions f, q and the ideas and techniques in Pazy [13].

The nonlocal Cauchy problem for functional differential equations with delay is also studied by Byszewski. In the paper [4], Byszewski discuss the existence, uniqueness and continuous dependence on initial data of solutions to the nonlocal Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x_t), \qquad t \in (\sigma, T], \\ x_\sigma &= \varphi + q(x_{t_1}, x_{t_2}, x_{t_3}, ..., x_{t_n}), \end{aligned}$$

where A is the infinitesimal generator of a C_0-semigroup of linear operators; $t_i \in [\sigma, T]$; $x_t \in C([-r, 0] : X)$ and $q : C([-r, 0] : X)^n \to X$, $f : [\sigma, T] \times C([-r, 0] : X) \to X$ are appropriate functions.

For a complementary bibliography for differential equations with nonlocal conditions, refer to [1]-[4], [12] and the references contained therein.

On the other hand, the neutral partial functional differential equation with unbounded delay

$$\begin{aligned} \tfrac{d}{dt}(x(t) + F(t, x_t)) &= Ax(t) + G(t, x_t), \qquad t \geq \sigma, \\ x_\sigma &= \varphi \in \mathcal{B}, \end{aligned} \tag{3}$$

is studied by Hernández & Henríquez in [9]-[10]. Using classical point fixed theorems and the ideas in [13], the authors establish the existence of mild, strong and periodic solutions for (3).

The results in this paper are continuation and generalization of the results reported by Byszewski in [1]-[5] and Hernández & Henríquez [10]. In particular, the existence of classical solutions for system (3), a non considered problem in [10], will be consequence of Theorem 2.3.

Throughout this paper, X will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \to X$ will be the infinitesimal generator of an analytic semigroup of linear operators, $T = (T(t))_{t \geq 0}$, on X. For the theory of strongly continuous semigroup, refer to Pazy [13] and Goldstein [6]. We will point out here some notations and properties that will be used in this work. It is well know that there exist constants $\tilde{M}$ and $w \in \mathbb{R}$ such that

$$\| T(t) \| \leq \tilde{M} e^{wt}, \qquad t \geq 0.$$

If T is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|,$$

define a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\| \cdot \|_\alpha$, then the following properties are well known (see [13]):

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LEMMA 1.1. *Assume that the previous condition hold.*

1) Let $0 < \alpha \leq 1$. *Then* X_α *is a Banach space.*
2) If $0 < \beta \leq \alpha$ *then* $X_\alpha \to X_\beta$ *is continuous.*
3) For every constant $a > 0$ *there exists* $C_a > 0$ *such that*

$$\| (-A)^\alpha T(t) \| \leq \frac{C_a}{t^\alpha}, \qquad 0 < t \leq a.$$

4) For every $a > 0$ *there exists a positive constant* C'_a *such that*

$$\| (T(t) - I)(-A)^{-\alpha} \| \leq C'_a t^\alpha, \qquad 0 < t \leq a.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [7]. To establish the axioms of the space $\mathcal{B}$ we follow the terminology used in Hino, Murakami & Naito [11]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(**A**) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

i) x_t is in $\mathcal{B}$.
ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.
iii) $\| x_t \| \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(**B**) The space $\mathcal{B}$ is complete.

For the literature on phase space, refer to Hino [11]. While noting here that from the axiom (**A1**), it follows that the operator function $W(\cdot)$, given by

$$[W(t)\varphi](\theta) := \begin{cases} T(t + \theta)\varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t + \theta) & \text{for } -\infty < \theta < -t, \end{cases} \tag{4}$$

is a strongly continuous semigroup of linear operators on $\mathcal{B}$. We will use the notation $A_W : D(A_W) \to X$ for the infinitesimal generator of $W(\cdot)$.

To obtain some of our results, we will require additional properties for the phase space $\mathcal{B}$, in particular we consider the following axiom (see [8] pp. 526, for details);

(**C$_3$**) Let $a > 0$. Let $x : (-\infty, \sigma + a] \to X$ be a continuous function such that $x_\sigma \equiv 0$ and the right derivative, denoted $\dot{x}(0^+)$, exists. If the function ψ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) := \dot{x}(0^+)$ belongs to $\mathcal{B}$, then $\| (\frac{1}{h})x_h - \psi \|_{\mathcal{B}} \to 0$ as $h \to 0^+$.

This paper contain two sections. In section 2 we define the different concepts used in this work and discuss the existence of mild, strong and classical solutions of the abstract nonlocal Cauchy problem (1). In general, the results are obtained using the contraction mapping principle and the techniques in Pazy [13] and Hernández & Henriquez [10].

The terminology and notations are those generally used in operator theory. In particular, if Z and Y are Banach spaces, we indicate by $\mathcal{L}(Z:Y)$ the Banach space of the bounded linear operators from Z into Y and we abbreviate to $\mathcal{L}(Z)$ whenever $Z=Y$. In addition $B_r[x:Z]$ will denote the closed ball in Z with center at x and radius r.

For a nonnegative bounded function $\xi:[\sigma,T]\to\mathbb{R}$ and $\sigma\le t\le T$ we will employ the notation ξ_t for

$$\xi_t=\sup\{\xi(\theta):\theta\in[\sigma,t]\}.$$

For $x\in X$, we will use the notation $\mathcal{X}_x$ for the function $\mathcal{X}_x:(-\infty,0]\to X$ where $\mathcal{X}_x(\theta)=0$ for $\theta<0$ and $\chi_x(0)=x$.

Finally, for a differentiable function $j:[0,T]\times\mathcal{B}\to X$ we will employ the decompositions:

$$j(s,\psi)-j(t,\psi) \;=\; D_1j(t,\psi)(s-t)+W_1(j,t,s,\psi) \tag{5}$$

$$j(t,\psi_1)-j(t,\psi) \;=\; D_2j(t,\psi)(\psi_1-\psi)+W_2(j,t,\psi,\psi_1) \tag{6}$$

where

$$\frac{W_1(j,t,s,\psi)}{\mid t-s\mid} \;\to\; 0, \qquad as \quad s\to t,$$

$$\frac{W_1(j,t,\psi,\psi_1)}{\parallel \psi-\psi_1\parallel_{\mathcal{B}}} \;\to\; 0, \qquad as \quad \psi_1\to\psi.$$

2. EXISTENCE RESULTS

In this section we discuss the existence of mild and regular solutions of the abstract nonlocal Cauchy problem (1). In order to define the concept of mild solution, we associate to problem (2) the integral equation

$$\begin{aligned} u(t) \;=\; & T(t-\sigma)(\varphi(0)+F(\sigma,u_0)+h(u_{t_1},u_{t_2},u_{t_3},...,u_{t_n})(0)\,)-F(t,u_t) \\ & -\int_\sigma^t AT(t-s)F(s,u_s)ds+\int_\sigma^t T(t-s)G(s,u_s)ds. \end{aligned} \tag{7}$$

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Definition 2.1. A function $u : (-\infty, T] \to X$, is a mild solution of the abstract Cauchy problem (3) if: $u_\sigma = \varphi + h(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n})$; the restriction of $u(\cdot)$ to the interval $[\sigma, T]$ is continuous; for each $\sigma \leq t \leq T$ the function $AT(t-s)F(s, u_s)$, $s \in [\sigma, t)$, is integrable and (7) hold on $[\sigma, T]$.

Definition 2.2. A function $u : (-\infty, T] \to X$, is a strong solution of the abstract nonlocal Cauchy problem (1) if: $u_\sigma = \varphi + h(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n})$; $u(t)$ and $F(t, u_t)$ are differentiable a.e. on $[\sigma, T]$; $u'(t)$ and $F'(t, u_t)$ belong to $\mathcal{L}^1([\sigma, T] : X)$ and $u(\cdot)$ satisfies a.e. (1) on $[\sigma, T]$.

Definition 2.3. A function $u : (-\infty, T] \to X$, is a classical solution of the abstract nonlocal Cauchy problem (1) if: $u_\sigma = \varphi + h(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n})$, the restriction of $u(\cdot)$ to the interval $[\sigma, T]$ is continuous, $u(t) \in D(A)$ for all $t \in (\sigma, T)$, $\dot{u}$ is continuous on (σ, T) and $u(\cdot)$ satisfies equation (1) on (σ, T).

Next, we show the existence and uniqueness of mild solutions using the contraction mapping principle, for this reason we introduce the following technical assumptions.

Assumptions : There exist $r > 0$ such that $B_r[0, \mathcal{B}] \subset \Omega$ and

$(\mathbf{H_1})$ $q : \mathcal{B}^n \to \mathcal{B}$ is continuous and exist positive constants $L_i(q)$ such that

$$\| q(\psi_1, \psi_2, \psi_3, ..., \psi_n) - q(\varphi_1, \varphi_2, \varphi_3, ..., \varphi_n) \| \leq \sum_{i=1}^{n} L_i(q) \| \psi_i - \varphi_i \|_{\mathcal{B}}$$

for every $\psi_i, \varphi_i \in B_r[0, \mathcal{B}]$.

$(\mathbf{H_2})$ There exist $\beta \in (0, 1)$ and positive constants $L(G)$, $L_\beta(F)$ such that; F is a X_β-valued function and the following Lipschitz conditions hold

$$\| (-A)^\beta F(t, \psi_1) - (-A)^\beta F(s, \psi_2) \| \leq L_\beta(F)(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}),$$

$$\| G(t, \psi_1) - G(s, \psi_2) \| \leq L(G)(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}})$$

for every $\sigma \leq s, t \leq T$ and $\psi_1, \psi_2 \in B_r[0, \mathcal{B}]$.

Remark 2. 1. In the rest of this work, to simplify notations, we will assume that $\sigma = \mathbf{0}$ and that $M_T \geq 1$.

Remark 2. 2. In the next result we will use the notations N_q for

$$N_q = \sup\{\| q(\psi_1, \psi_2, \psi_3, ..., \psi_n) \| : \psi_i \in B_r[0, \mathcal{B}]\}$$

and $N_{(-A)^\beta F}$, N_F, N_G for the sup of $(-A)^\beta F$, F and G on $[0, T] \times B_r[0, \mathcal{B}]$.

2.1 Mild Solution

THEOREM 2.1. *Let assumptions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *be satisfied. Let* $\varphi \in B_r(0,\mathcal{B})$ *and assume that*

$$\Theta = \max\left\{K_T\Lambda_1 + M_T^2\sum_{i=1}^{n} L_i(q),\ \Lambda_2 + \sum_{i=1}^{n} L_i(q)M_T\right\} < 1, \tag{8}$$

$$\begin{aligned}\rho &= (\tilde{M}HK_T + M_T)\parallel \varphi \parallel + N_q M_T \\ &\quad + K_T\left(N_q\tilde{M}H + N_F(\tilde{M}+1) + \frac{C_{1-\beta}}{\beta}N_{(-A)^\beta F}T^{1-\beta} + \tilde{M}N_G T\right) < r,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\Lambda_1 &= M_T\left(\tilde{M}H\sum_{i=1}^{n} L_i(q) + \parallel (-A)^{-\beta}\parallel L_\beta(F) + \frac{C_{1-\beta}}{\beta}L_\beta(F)T^\beta + \tilde{M}L(G)T\right) \\ &\quad + \tilde{M}\parallel (-A)^{-\beta}\parallel L_\beta(F), \\ \Lambda_2 &= K_T\left(\tilde{M}H\sum_{i=1}^{n} L_i(q) + \parallel (-A)^{-\beta}\parallel L_\beta(F) + \frac{C_{1-\beta}L_\beta(F)}{\beta}T^\beta + \tilde{M}L(G)T\right).\end{aligned}$$

Then there exists a unique mild solution $u(\cdot,\varphi)$ *of the nonlocal problem (1).*

Proof: Let $\mathcal{B}C([0,T]:X)$ be the space

$$\mathcal{B}C([0,T]:X) = \{u : (-\infty,T] \to X :\ u_0 \in \mathcal{B};\ u \in C([0,T]:X)\}$$

endowed with the norm $\parallel u \parallel_{\mathcal{B}C} = M_T \parallel u_0 \parallel_{\mathcal{B}} + K_T \parallel u \parallel_T$. On $Y = B_r[0,\mathcal{B}C([0,T]:X)]$ we define the mapping $\Upsilon : Y \to \mathcal{B}C([0,T]:X)$ by

$$\begin{aligned}\Upsilon u(t) &= T(t)(\varphi(0) + F(0,u_0) + q(u_{t_1},u_{t_2},u_{t_3},...,u_{t_n})(0)\,) - F(t,u_t) \\ &\quad + \int_0^t (-A)^{1-\beta}T(t-s)(-A)^\beta F(s,u_s)ds + \int_0^t T(t-s)G(s,u_s)ds, \qquad t\in[0,T], \\ (\Upsilon u)_0 &= \varphi + q(u_{t_1},u_{t_2},u_{t_3},...,u_{t_n}).\end{aligned}$$

From the proof of Theorem 2.1 in [10], follow that the mapping Υ is well defined. Next we will prove that $\Upsilon(Y) \subset Y$ and that Υ is a contraction on Y. Let $u \in Y$ and $t \in [0,T]$. From axiom $(\mathbf{A})$, $u_t \in B_r[0,\mathcal{B}]$ and in this case

$$\parallel \Upsilon u(t) \parallel \le \tilde{M}(H(\parallel \varphi \parallel_{\mathcal{B}} + N_q) + N_F) + N_F + \int_0^t \left(\frac{C_{1-\beta}N_{(-A)^\beta F}}{(t-s)^{1-\beta}} + \tilde{M}N_G\right) ds, \tag{10}$$

and

$$\| (\Upsilon u)_0 \| \ \leq \ \| \varphi \|_{\mathcal{B}} + N_q. \tag{11}$$

From (10)-(11), (8) and the definition of $\| \cdot \|_{\mathcal{B}C}$, we have that $\Upsilon(u) \in Y$.

In order to prove that Υ satisfies a Lipschitz condition, we take $u, v \in Y$. If $t \in [0,T]$, from axiom (**A**) we see that can to compute

$$\begin{aligned} \| \Upsilon u(t) - \Upsilon v(t) \| \ &\leq \ \tilde{M} \left(H \sum_{i=1}^{n} L_i(q) \| u_{t_i} - v_{t_i} \|_{\mathcal{B}} + L_\beta(F) \| (-A)^{-\beta} \| \| u_0 - v_0 \|_{\mathcal{B}} \right) \\ &\quad + \| (-A)^{-\beta} \| L_\beta(F) \| u_t - v_t \|_{\mathcal{B}} \\ &\quad + \int_0^t \frac{C_{1-\beta} L_\beta(F)}{(t-s)^{1-\beta}} \| u_s - v_s \|_{\mathcal{B}} \, ds + \int_0^t \tilde{M} L(G) \| u_s - v_s \|_{\mathcal{B}} \, ds \\ &\leq \ \Lambda_1 \| u_0 - v_0 \|_{\mathcal{B}} + \Lambda_2 \| u - v \|_T, \end{aligned}$$

thus

$$\| \Upsilon u - \Upsilon v \|_T \leq \Lambda_1 \| u_0 - v_0 \|_{\mathcal{B}} + \Lambda_2 \| u - v \|_T \ . \tag{12}$$

On the other hand, a simple calculus prove that

$$\| (\Upsilon u)_0 - (\Upsilon v)_0 \|_{\mathcal{B}} \leq \sum_{i=1}^{n} L_i(q) M_T \| u_0 - v_0 \|_{\mathcal{B}} + \sum_{i=1}^{n} L_i(q) K_T \| u - v \|_T \ . \tag{13}$$

Finally, from (12)-(13) and assumption (8) we infer that Υ is a contraction on Y. Clearly a fixed point of Υ is the unique mild solution of the nonlocal problem (1). The proof is complete.

2.2 Strong solution

The proof of the existence of strong solutions of the problem (1) follows from the steeps in the proofs of Lemma 3.1 and Theorem 3.1 in [10], for this reason we will omit the proofs of the following results.

Proposition 2.1. *Assume that assumptions in Theorem 2.1 hold. Let $u(\cdot) = u(\cdot, \varphi)$ be the mild solution of (1). If $t \to W(t)(\varphi + q(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n}))$ is Lipschitz on $[0,T]$ and $F(0, u_0) \in D(A)$ then $t \to u_t(\cdot, \varphi)$ is Lipschitz continuous on $[0,T]$.*

Lemma 2.1. *Assume that assumptions in Theorem 2.1 hold. Let $u(\cdot) = u(\cdot, \varphi)$ be the mild solution of (1) and $v, w : [0,T] \to X$ the functions*

$$v(t) = \int_0^t AT(t-s) F(s, u_s) ds, \quad w(t) = \int_0^t T(t-s) G(s, u_s) ds.$$

If v, w are differentiable a.e on $[0,T]$ and $t \to AF(t,u_t) \in \mathcal{L}^1([0,T]:X)$, then $(v(t), w(t)) \in D(A) \times D(A)$ a.e. $t \in [0,T]$.

THEOREM 2.2. *Let assumption in Theorem 2.1 be satisfied. Let $u(\cdot) = u(\cdot, \varphi)$ be the mild solution of (1) and assume that the following conditions hold:*

1.the function $t \to W(t)(\varphi + q(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n}))$ is Lipschitz on $[0,T]$,
2.$F(0,u_0) \in D(A)$ and $t \to AF(t,u_t) \in \mathcal{L}^1([0,T]:X)$,
3.X is a reflexive space.

Then $u(\cdot, \varphi)$ is a strong solution.

2.3 Classical solution

In the proof of the next result, by slight abuse of notation, and in order to abbreviate the notations, we denote by $\|\cdot\|$ the norm of $\mathcal{L}(X)$. Now, we establish the principal result of this work.

THEOREM 2.3. *Assume that assumptions in Theorem 2.1 hold. Let $u(\cdot) = u(\cdot, \varphi)$ be the mild solution of (1) and assume, furthermore, that*

1.the functions $(-A)^\beta F$ and G are continuously differentiable and $t \to AF(t,u_t)$ is continuous on $[0,T]$,

2.$u_0 \in D(A_W)$, $D_2F(0,u_0) \equiv 0$ and

$$\mu \;=\; K_T[\| D_2F(t,u_t) \|_T + \frac{C_{1-\beta}}{\beta} \| D_2(-A)^\beta F(t,u_t) \|_T T^\beta + \tilde{M} \| D_2G(t,u_t) \|_T T] < 1. \tag{14}$$

If $\mathcal{X}_{G(0,u_0)} \equiv 0$ or $\mathcal{X}_{G(0,u_0)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$ then $u(\cdot, \varphi)$ is a classical solution.

Proof. Let $z(\cdot) : (-\infty, T) \to X$ be a solution of the integral equation

$$\begin{aligned} z(t) \;&=\; T(t)(Au(0) + G(0,u_0)) + p(t) - D_2F(t,u_t) \cdot z_t \\ &\quad + \int_0^t (-A)^{1-\beta} T(t-s) D_2(-A)^\beta F(s,u_s) \cdot z_s ds \\ &\quad + \int_0^t T(t-s) D_2G(s,u_s) \cdot z_s ds, \\ z_0 \;&=\; A_W(u_0) + \mathcal{X}_{G(0,u_0)}, \end{aligned} \tag{15}$$

where

$$p(t) \;=\; -D_1F(t,u_t) + \int_0^t T(t-s) \left((-A)^{1-\beta} D_1(-A)^\beta F(s,u_s) + D_1G(s,u_s) \right) ds.$$

The existence and uniqueness of solution for the system (15)-(15) is consequence of the contraction mapping principle and condition (13). In what follow, we prove that $\dot{u}(\cdot) = z(\cdot)$. Using the decomposition introduced in (5)-(6), for $t \in [0, T)$ and $h > 0$ sufficiently small, we get

$$
\begin{aligned}
&\| \xi(t,h) \| = \| \frac{u(t+h)-u(t)}{h} - z(t) \| \leq \| T(t)(\frac{T(h)-I}{h}u(0) - Au(0)\,) \| \\
&+ \| T(t)(\frac{T(h)-I}{h})F(0,u_0) + \frac{1}{h}\int_0^h (-A)^{1-\beta}T(t+h-s)(-A)^{\beta}F(s,u_s)ds \| \\
&+ \| D_1F(t,u_{t+h}) - D_1F(t,u_t) \| + \| D_2F(t,u_t)\cdot\xi_t(\cdot,h) \| \\
&+ \| \frac{W_1(F,t,t+h,u_{t+h})}{h} \| + \| \frac{W_2(F,t,u_t,u_{t+h})}{h} \| \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_1(-A)^{\beta}F(s,u_{s+h}) - D_1(-A)^{\beta}F(s,u_s) \| ds \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^{\beta}F(s,u_s)\cdot\xi_s(\cdot,h) \| ds \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left(\| \frac{W_1((-A)^{\beta}F,s,s+h,u_{s+h})}{h} \| + \| \frac{W_2((-A)^{\beta}F,s,u_s,u_{s+h})}{h} \| \right) ds \\
&+ \| \frac{1}{h}\int_0^h T(t+h-s)G(s,u_s)ds - T(t)G(0,u_0) \| \\
&+ \int_0^t \tilde{M} \| D_1G(s,u_{s+h}) - D_1G(s,u_s) \| ds + \int_0^t \tilde{M} \| D_2G(s,u_s)\cdot\xi_s(\cdot,h) \| ds \\
&+ \tilde{M}\int_0^t (\| \frac{W_1(G,s,s+h,u_{s+h})}{h} \| + \| \frac{W_2(G,s,u_s,u_{s+h})}{h} \|)ds.
\end{aligned}
$$

Since $u_0 \in D(A_W)$ and $t \to AF(t,u_t)$ is continuous, we know from Proposition (2.1) that $t \to u_t(\cdot,\varphi)$ is Lipschitz continuous, thus

$$
\frac{1}{h} \| W_2(G,s,u_s,u_{s+h}) \| = \frac{\| W_2(G,s,u_s,u_{s+h}) \|}{\| u_{s+h} - u_s \|_{\mathcal{B}}} \cdot \frac{\| u_{s+h} - u_s \|_{\mathcal{B}}}{h} \to 0, \tag{16}
$$

and

$$
\frac{1}{h} \| W_2((-A)^{\beta}F,s,u_s,u_{s+h}) \| = \frac{\| W_2((-A)^{\beta}F,s,u_s,u_{s+h}) \|}{\| u_{s+h} - u_s \|_{\mathcal{B}}} \cdot \frac{\| u_{s+h} - u_s \|_{\mathcal{B}}}{h} \to 0 \tag{17}
$$

uniformly for $s \in [0,T)$. Using (16) and (17), we can to rewrite the previous inequality in the form

$$\| \xi(t,h) \| \leq \rho(t,h) + \frac{1}{h}\int_0^h \| (-A)^{1-\beta}T(t+h-s)\left((-A)^{\beta}F(s,u_s) - (-A)^{\beta}F(0,u_0)\right) \| \, ds$$
$$+ \| D_2F(t,u_t) \| \| \xi_t(\cdot,h) \| + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^{\beta}F(s,u_s) \| \| \xi_s(\cdot,h) \|_{\mathcal{B}} \, ds$$
$$+\tilde{M}\int_0^t \| D_2G(s,u_s) \| \| \xi_s(\cdot,h) \|_{\mathcal{B}} \, ds,$$

where $\rho(t,h) \to 0$ as $h \to 0$, uniformly for $t \in [0,T)$. Using axiom $(\mathbf{A})$, assumption (13) and the Lipschiz continuity of $t \to u_t$, from the last inequality we have that

$$\| \xi(\cdot,h) \|_t \leq \frac{\rho(t,h)}{1-\mu} + \frac{C_{1-\beta}L_{\beta}(F)Ch^{\beta}}{(1-\mu)\beta} + \frac{\mu M_T}{(1-\mu)K_T} \| \frac{u_h - u_0}{h} - z_0 \|_{\mathcal{B}} . \tag{18}$$

Next, we prove that $\| \frac{u_h-u_0}{h} - z_0 \|_{\mathcal{B}}$ as $h \to 0$. To show this convergence we introduce the decomposition $u(\cdot) = y(\cdot) + \sum_{i=1}^3 z^i$ where

$$\begin{aligned} z^1(\theta) &= T(\theta)F(0,u_0) - F(\theta,u_\theta), \\ z^2(\theta) &= \int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^{\beta}F(s,u_s)ds, \\ z^3(\theta) &= \int_0^\theta T(\theta-s)G(s,u_s)ds, \end{aligned}$$

for $\theta \in [0,T)$ and $z_0^i = 0$ for every $i \in \{1,2,3\}$. According the above notations we see

$$\| \frac{u_h - u_0}{h} - A_W(u_0) - \mathcal{X}_{G(0,u_0)} \|_{\mathcal{B}} \leq \| \frac{W(h)u_0 - u_0}{h} - A_W(u_0) \|_{\mathcal{B}} + \| \frac{z_h^1 + z_h^2}{h} \|_{\mathcal{B}}$$
$$+ \| \frac{z_h^3}{h} - \mathcal{X}_{G(0,u_0)} \|_{\mathcal{B}} . \tag{19}$$

Now, we estimate each term of (19) separately. Clearly

$$I_1(h) \to 0 \qquad \text{as} \qquad h \to 0. \tag{20}$$

On the other hand, if either, $\mathcal{X}_{G(0,u_0)} \equiv 0$ or $\mathcal{X}_{G(0,u_0)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$

$$I_3(h) \to 0 \qquad \text{as} \qquad h \to 0. \tag{21}$$

In relation with the second term in the right side of (19), from Axiom $(\mathbf{A})$ we find

$$I_2(h) \leq K_T\frac{1}{h} \| (T(\theta) - I)F(0,u_0) + \int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^{\beta}F(s,u_s)ds \|_h \, ds$$
$$+K_T\frac{1}{h} \| F(0,u_0) - F(\theta,u_\theta) \|_h . \tag{22}$$

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From the Lipschitz continuity of $s \to x_s$, for $\theta \in [0,h]$ we have that

$$\begin{aligned}\frac{1}{h} \ \| \ (T(\theta)-I)F(0,u_0) + \int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,u_s)ds \ \| \\ \leq \ \frac{1}{h}\int_0^\theta \| \ (-A)^{1-\beta}T(\theta-s)\{(-A)^\beta F(s,u_s)-(-A)^\beta F(0,u_0)\} \ \| \ ds \\ \leq \ \frac{1}{h}\int_0^\theta \frac{C_{1-\beta}L_\beta(F)}{(\theta-s)^{1-\beta}}Chds \leq C_{1-\beta}L_\beta(F)C\frac{h^\beta}{\beta},\end{aligned}$$

thus,

$$\frac{1}{h} \ \| \ (T(\theta)-I)F(0,u_0) + \int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,u_s)ds \ \|_h \to 0 \tag{23}$$

as $h \to 0$.

On the other hand

$$\| \ \frac{F(0,u_0)-F(\theta,u_\theta)}{\theta} \cdot \frac{\theta}{h} \ \|_h \to 0 \tag{24}$$

as $h \to 0$, since $DF(0,u_0) \equiv 0$. Using (23) and (24) in (22), we conclude that

$$I_2(h) \to 0 \quad \text{as} \quad h \to 0. \tag{25}$$

Inequalities (18)-(21) and (25) implies that $\frac{u_h-u_0}{h} \to z_0$ as $h \to 0$ and thus, $\dot{u}(\cdot) = z(\cdot)$.

From Theorem 2.4.1 in [6] and Lemma 2.2 below, we have that $u(t) \in D(A)$ for $t \in [0,T)$ and that $\dot{u} \in C((0,T):X)$. Finally, from the proof of Theorem 3.1 in [10] we conclude that $u(\cdot)$ is solution of (1). The proof is complete.

The proof of the next result is analogous to the proof of Theorem 2.4.1 in [6]. However there are some differences that require attention and we include the principal ideas of the proof for completeness.

Lemma 2.2. *Let $0 < \beta < 1$, $g \in C([0,T]:X_{1-\beta}) \bigcap C^{0,\vartheta}([0,T]:X)$ and $y:[0,T] \to X$ the function defined by*

$$y(t) = \int_0^t (-A)^{1-\beta}T(t-s)g(s)ds.$$

If $\beta + \vartheta > 1$, then $y(t) \in D(A)$ for every $t \in [0,T]$ and $\dot{y} \in C((0,T):X)$.

Proof. For $t \in [0,T)$ we rewrite $y(t)$ in the form

$$\int_0^t (-A)^{1-\beta}T(t-s)(g(s)-g(t))ds + \int_0^t (-A)^{1-\beta}T(t-s)g(t)ds = v(t)+w(t). \tag{26}$$

Clearly, $Aw(t) = T(t)(-A)^{1-\beta}g(t) - (-A)^{1-\beta}g(t) \in C([0,T]:X)$.

For $\epsilon > 0$ sufficiently small we define the function

$$v_\varepsilon(t) := \begin{cases} \int_0^{t-\epsilon}(-A)^{1-\beta}T(t-s)(g(s)-g(t))ds\,, & \text{for } t \in [\epsilon, T), \\ 0 & \text{for } t \in [0, \epsilon]. \end{cases} \tag{27}$$

It is clear that $v_\epsilon(t) \in D(A)$. Moreover for $0 < \delta_1 < \delta_2$

$$\begin{aligned} \| Av_{\delta_2}(t) - Av_{\delta_1}(t) \| &\leq \int_{t-\delta_2}^{t-\delta_1} \| (-A)^{2-\beta}T(t-s)(g(s)-g(t)) \| ds \\ &\leq \int_{t-\delta_2}^{t-\delta_1} \tfrac{C_{2-\beta}}{(t-s)^{2-\beta}}|s-t|^\vartheta ds \leq C_{2-\beta}(\delta_2^{\beta+\vartheta-1} - \delta_1^{\beta+\vartheta-1}). \end{aligned}$$

The last inequality proves that $A(v_\delta)(t)$ is convergent, $\beta + \vartheta > 1$, and since A is a closed operator

$$A(v(t)) = \int_0^t A^{2-\beta}T(t-s)(g(s)-g(t))ds,$$

thus, $y(t) \in D(A)$.

The continuity of $\partial_t y$ follows as in [6], Theorem 2.4.1. The proof is finished.

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