Singularities of Matrices and Determinantal Varieties

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Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.
3. Invariants of Determinantal Singularities.
4. Nash transformation of an EIDS
5. Sections of EIDS.
6. Euler obstruction of EIDS.

Recent PhD. Thesis on Determinantal Varieties.

- Miriam Silva Pereira, ICMC, 2010. http://www.teses.usp.br/teses/disponiveis/55/55135/tde-22062010-133339/pt-br.php
- Brian Pike, Northeastern University, 2010. http://www.brianpike.info/thesis.pdf
- Bruna Oréfice Okamoto, UFSCar, 2011 http://www.dm.ufscar.br/ppgm/attachments/article/179/download.pdf
- Nancy Carolina Chachapoyas Siesquén, ICMC and Université Aix Marseille, 2014. http://www.teses.usp.br/teses/disponiveis/55/55135/tde-13022015-100258/fr.php

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• W. Ebeling e S. M. Gusein-Zade, *On indices of* 1*-forms on determinantal singularities*, *Singularities and Applications*, **267**, 119-131, (2009).

Generic determinantal variety

Definition

Let $M_{m,n}$ be the set of all $m \times n$ matrices with complex entries, and for all $t \leq \min\{m, n\}$ let

$$M^t_{m,n} = \{A \in M_{m,n} | rank(A) < t\}.$$

This set is a singular variety, called generic determinantal variety.

1. $M^t_{m,n}$ has codimension $(n - t + 1)(m - t + 1)$ in $M_{m,n}$
2. The singular set of $M^t_{m,n}$ is $M^{t-1}_{m,n}$
3. $M^t_{m,n} = \cup_{i=1,\dots,t}(M^i_{m,n} \backslash M^{i-1}_{m,n})$, this partition is a Whitney stratification of $M^t_{m,n}$.

Determinantal varieties

Let $F : U \subset \mathbb{C}^N \to M_{m,n}$. For each x, $F(x) = (f_{ij}(x))$ is a $m \times n$ matrix; the coordinates f_{ij} are complex analytic functions on U.

Definition

A determinantal variety of type (m, n, t), in an open domain $U \subset \mathbb{C}^N$ is a variety X that satisfies:

- *X is the preimage of the variety $M^t_{m,n}$. That is $X = F^{-1}(M^t_{m,n})$.*
- *$codim(X) = (m - t + 1)(n - t + 1)$ in $\mathbb{C}^N$*

Determinantal varieties

Example

Determinantal surface Let F be the following map:

$$\begin{array}{rcccl} F & : & \mathbb{C}^4 & \to & M_{2,3} \\ & & (x,y,z,w) & \mapsto & \begin{pmatrix} z & y & x \\ w & x & y \end{pmatrix} \end{array}$$

Then $X = F^{-1}(M^2_{2,3}) = V(zx - wy, zy - wx, y^2 - x^2)$, X is a surface in $\mathbb{C}^4$ with isolated singularity at the origin.

Essentially Isolated Determinantal Singularities (EIDS)

The *Essential Isolated Determinantal Singularities (EIDS)* were defined by Ebeling and Gusein-Zade in [Proc. Steklov Inst. Math. (2009)].

Definition EIDS:

A germ $(X,0) \subset (\mathbb{C}^N,0)$ of a determinantal variety of type (m,n,t) has an essentially isolated determinantal singularity at the origin (EIDS) if F is transverse to all strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the stratification of $M^t_{m,n}$ in a punctured neighbourhood of the origin.

The singular set of an EIDS $X = F^{-1}(M^t_{m,n})$ is the EIDS $F^{-1}(M^{t-1}_{m,n})$.

We can suppose $(X, 0)$, of type (m, n, t) is defined by $F : (\mathbb{C}^N, 0) \to M_{m,n}$ with $F(0) = 0$, because if $F(0) \neq 0$ and therefore rank $F(0) = s > 0$, then $(X, 0)$ is a determinantal singularity of type $(m-s, n-s, t-s)$ defined by $F' : (\mathbb{C}^N, 0) \to M_{m-s,n-s}$, with $F'(0) = 0$.

Example

An ICIS is an EIDS of type $(1, n, 1)$

Example

The determinantal variety represented by the matrix

$$N = \begin{pmatrix} z & y & x \\ 0 & x & y \end{pmatrix}$$

is a curve in $\mathbb{C}^3$.

More generally, $n \times (n+1)$ matrices with entries in $\mathcal{O}_N$ give a presentation of Cohen-Macaulay varieties of codimension 2 (Hilbert-Burch theorem).

Deformations of EIDS

Deformations (in particular, smoothings) of determinantal singularities which are themselves determinantal ones.

Definition

(Ebeling and Gusein Zade (2009)) *An* *essential smoothing* $\tilde{X}$ *of the EIDS* $(X, 0)$ *is a subvariety lying in a neighbourhood* U *of the origin in* $\mathbb{C}^N$ *and defined by a perturbation* $\tilde{F} : U \to M_{m,n}$ *of the germ* F *such that* $\tilde{F}$ *is transversal to all the strata* $M^i_{m,n} \setminus M^{i-1}_{m,n}$*, with* $i \leq t$*.*

Example

For generic values of a, b, c, $\tilde{N}$ gives a smoothing of the curve in $\mathbb{C}^3$.

$$\tilde{N} = \begin{pmatrix} z & y+a & x+b \\ c & x & y \end{pmatrix}$$

Isolated determinantal singularities (IDS)

Proposition

- *An EIDS* $(X,0) \subset (\mathbb{C}^N,0)$ *of type* (m,n,t), *defined by* $F:(\mathbb{C}^N,0) \to (M_{m,n},0)$ *has an isolated singularity at the origin if and only if* $N \leq (m-t+2)(n-t+2)$.
- $(X,0)$ *has a smoothing if and only if* $N < (m-t+2)(n-t+2)$.

Example

$F:\mathbb{C}^N \to M_{2,3},\ F \pitchfork M^i_{2,3},\ i=1,2.$

$$F(x) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

When $N=6$, the singularity of $X = F^{-1}(M^2_{2,3})$ is isolated and X has no smoothing.

Matrices Singularity Theory

The methods of singularity theory apply both to real and complex matrices.

- Arnol'd (1971), square matrices.
- Bruce (2003), simple singularities of symmetric matrices.
- Bruce and Tari (2004), simple singularities of square matrices.
- Haslinger (2001), simple skew-symmetric.
- Frühbis-Krüger (2000) and Frühbis-Krüger and Neumer (2010), Cohen-Macaulay codimension 2 simple singularities.
- Goryunov and Mond (2005), Tjurina and Milnor numbers of square matrices.
- M. Silva Pereira (2010), singularity theory of general $n \times m$ matrices.

Let $\mathcal{R}$ be the group of changes of coordinates in the source $(\mathbb{K}^N, 0)$, that is,

$$\mathcal{R} = \{h : (\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0), \text{germs of analytic diffeomorphisms}\}$$

We denote by $\mathcal{H} = GL_m(\mathcal{O}_N) \times GL_n(\mathcal{O}_N)$ and $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ (semi-direct product)

Definition

Given two matrices $F_1(x) = (f^1_{ij}(x))_{m\times n}$ and $F_2(x) = (f^2_{ij}(x))_{m\times n}$, we say that

$$F_1 \sim F_2 \text{ if } \exists\ (\phi, R, L) \in \mathcal{G} \text{ such that } F_1 = L^{-1}(\phi^* F_2)R.$$

Proposition

If $F_1 \sim F_2$ then the corresponding determinantal varieties $X_1^t = F_1^{-1}(M_{m,n}^t)$ and $X_2^t = F_2^{-1}(M_{m,n}^t)$, $1 \leq t \leq m$ are isomorphic.

Tangent Space to the $\mathcal{G}$-orbit of F.

We denote by $\Theta(F)$ the free $\mathcal{O}_N$ module of rank nm consisting of all deformations of $F : (\mathbb{C}^N, 0) \to M_{m,n}$.

The tangent space to the $\mathcal{G}$-orbit of F, $T\mathcal{G}(F) = T\mathcal{R}(F) + T\mathcal{H}(F)$.

$$T\mathcal{G}(F) = \mathcal{M}_N \left\{ \frac{\partial F}{\partial x_i} \right\} + \mathcal{O}_N \left\{ R_{lk}, C_{ij} \right\}$$

$$T_e\mathcal{G}(F) = \mathcal{O}_N \left\{ \frac{\partial F}{\partial x_i} \right\} + \mathcal{O}_N \left\{ R_{lk}, C_{ij} \right\}$$

where $C_{ij}(M)$ (respectively $R_{lk}(M)$) is the matrix which has the i-column (respectively l-row) equal to the j-column of M (respectively k-row) with zeros in other places.

The group $\mathcal{G}$ is a geometric subgroup of the contact group $\mathcal{K}$. Hence the infinitesimal methods of singularity theory applies.

Definition

$F : U \to M_{m,n}$ is $\mathcal{G}$-stable if $T_e\mathcal{G}(F) = \Theta(F)$.

The above condition holds if and only if F is transversal to the canonical stratification of the space $M_{m,n}$.

Definition

The germ $F : (\mathbb{C}^N, 0) \to M_{m,n},\ F(x) = (f_{ij}(x))$ *is* $k - \mathcal{G}$*-finitely determined if for every* $G : (\mathbb{C}^N, 0) \to M_{m,n},\ G(x) = (g_{ij}(x))$ *such that* $j^k f_{ij}(x) = j^k g_{ij}(x),\ 1 \leq i \leq m,\ 1 \leq j \leq n$, *then* $G \sim F$.

Theorem

(M.S. Pereira, PhD thesis) *F is $\mathcal{G}$-finitely determined if and only if the Tjurina number of F*

$$\tau(F) = dim_{\mathbb{K}} \frac{\Theta(F)}{T_e\mathcal{G}(F)}$$

is finite

In this case, F has a versal unfolding with τ-parameters.

Theorem

(M.S. Pereira, PhD thesis) *(Geometric criterion of finite determinacy) F is finitely $\mathcal{G}$-determined if and only if there exists a representative $F : U \to M_{m,n}$ such that for all $x \neq 0$ in U, $rank F(x) + 1 = i$, then F is transversal to $M^i_{m,n}$ at x.*

F is $\mathcal{G}$-finitely determined if and only if $X = F^{-1}(0)$ is an EIDS.

A stable perturbation $\tilde{F}$ of F defines an essential smoothing $\tilde{X} = \tilde{F}^{-1}(M^m_{m,n})$ of X.

Example

Let

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

This is the first normal form of the classification of simple Cohen-Macaulay singularities of codimension 2 of A. Fübhis-Kruger and A. Neumer in [Comm. Alg. 38, 454-495, (2010)].

The surface $X_k \subset \mathbb{C}^4$ associated to A_k is defined by the ideal $\langle xz^k - yw, x^2 - zw, xy - z^{k+1} \rangle$.

The versal unfolding of F_k is

$$\tilde{F}_k(x, y, z, w, u_0, u_1, \ldots, u_k) = \begin{pmatrix} x & y & z \\ w & z^k + \Sigma_0^{k-1} u_i z^i & x + u_k \end{pmatrix},$$

$$\tau(F_k) = k + 1.$$

Singular fibration of an EIDS

$$F : (\mathbb{C}^N, 0) \to M_{m,n}, \ (X, 0) = F^{-1}(M^t_{m,n})$$

$$\widetilde{F} : W \subset \mathbb{C}^N \times \mathbb{C}^s \to M_{m,n}, \widetilde{F}(x, 0) = F(x), \ \widetilde{F} \pitchfork \{M^i_{m,n} \backslash M^{i-1}_{m,n}\}, \ \mathfrak{X} = \widetilde{F}^{-1}(M$$

$$\begin{array}{ccc} \mathfrak{X} & \subset & W \subset \mathbb{C}^N \times \mathbb{C}^s \\ & & \downarrow \pi \\ B(F) & \subset & \mathbb{C}^s \end{array},$$

where $B(F)$ is the bifurcation set.

For $u \in \mathbb{C}^s \setminus B(F)$, $\widetilde{F}_u$ defines $\widetilde{X}_u$ which is an essential smoothing of X. The generic fibre $\widetilde{X}_u$ is well defined.

Invariants of EIDS

Definition

(Damon and Pike [Geom. Topol., **18**(2) (2014)], Ebeling and Gusein-Zade (2009)) *The singular vanishing Euler characteristic of X, is defined as*

$$\tilde{\chi}(X) = \tilde{\chi}(\widetilde{X}_u) = \chi(\widetilde{X}_u) - 1.$$

(Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J. Math. **197** (2013), 475-495.]) *When $\widetilde{X}_u$ is smooth, vanishing Euler characteristic of X is*

$$\nu(X) = (-1)^{dim(X)}(\chi(\widetilde{X}_u) - 1).$$

Let X and $\mathfrak{X}$ be as above, $\dim(X) = d$.

Definition: The d-polar multiplicity, (Gaffney [Top. (1993)])

Let $p : X \to \mathbb{C}$, with isolated singularity. Let

$$\pi : \mathfrak{X} \subset \mathbb{C}^N \times \mathbb{C} \to \mathbb{C},$$

$\pi^{-1}(0) = X$, $\widetilde{p} : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}$ linear projection, $\widetilde{p}(x,0) = p(x)$, and for all $t \neq 0, \widetilde{p}_t(.)$ is a Morse function.
Let

$$P_d(X, \pi, p) = \overline{\Sigma(\pi, \widetilde{p})|\mathfrak{X}_{reg}}$$

be the relative polar variety of X relative to π and p.
Define

$$m_d(X, \pi, p) = m_0(P_d(\pi, p)).$$

In general, $m_d(X, \pi, p)$ depends on the choices of $\mathfrak{X}$ and p, but when X is an EIDS, m_d depends only on X and p. Furthermore, if p is a generic linear embedding, m_d is an invariant of the EIDS X, denoted by $m_d(X)$.

Proposition:

Let $X = F^{-1}(M_{m,n})$ and $\widetilde{X}$ its essential smoothing. Let $p : X \to \mathbb{C}$ be a function with isolated singularity in X. Then

$$m_d(X) = \#\,\text{non-degenerated critical points of}\, \widetilde{p}_t|\widetilde{X}_t,$$

where $\widetilde{p}_t$ is a generic perturbation of p (Morsification), and $\widetilde{X}_t$. an essential smoothing of X.

Nash transformation

Let X be a $d-$ dimensional analytic complex variety in $\mathbb{C}^N$.

$Gr(d, N)$ the Grassmannian of d-subspaces in $\mathbb{C}^N$.

Let $\pi : \mathbb{C}^N \times Gr(d, N) \to \mathbb{C}^N$ be the projection to the $\mathbb{C}^N$.

On the regular part of X, we have the Gauss map defined by:

$$\begin{array}{rccc} s & : & X_{reg} & \to & \mathbb{C}^N \times Gr(d, N) \\ & & x & \mapsto & (x, T_x X_{reg}) \end{array}$$

Definition

The Nash transformation $\widehat{X}$ of X is the closure in $\mathbb{C}^N \times Gr(d, N)$ of the image of s, i.e.,

$$\widehat{X} = \overline{\{(x, W)|x \in X_{reg}, W = T_x X_{reg}\}}.$$

If $x \in X$ is a singular point, then the fibre over x:
$\nu^{-1}(x) = \{(x, T)/T = \lim_{x_n \to x}(T_{x_n}X), x_n \in X_{reg}\}, \ \nu = \pi|_{\widehat{X}}$

Proposition

(Arbarello, Cornalba, Griffiths and Harris)
The Nash transformation $\widehat{M^t_{m,n}}$ *of* $M^t_{m,n}$, $1 \leq t \leq m$ *is smooth.*

Theorem

(Chachapoyas-Siesquen, PhD thesis) *Let* $X = F^{-1}(M^t_{m,n}) \subset \mathbb{C}^N$ *be an EIDS, defined by* $F : U \subset \mathbb{C}^N \to M_{m,n}$.

If F *is transversal to all the limits of the tangent spaces to the strata of* $M^t_{m,n}$ *then* $\widehat{X}$ *is smooth.*

Question

Does a finite iteration of Nash transformations resolve the singularities of an EIDS X?

Isolated Determinantal Singularities admitting smoothing: N < (m-t+2)(n-t+2)

If $X \subset \mathbb{C}^N$ is a normal variety admitting smoothing, then $b_1(X_u) = 0$ (Greuel and Steenbrink [*Proc. Symp. Pure Math.* **40**,(1983).]) Determinantal isolated singularities are normal singularities, so this holds for them.

Theorem

(Nuno-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J. 2013]) *Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function and $\widetilde{X}$ an essential smoothing of X. Then*

$$\#\Sigma(p|_{\widetilde{X}}) = \nu(X, 0) + \nu(X \cap p^{-1}(0), 0),$$

where $\#\Sigma(p|_{\widetilde{X}})$ denotes the number of critical points of $p|_{\widetilde{X}}$.

Determinantal surfaces

M. S. Pereira and M. Ruas [Math. Scand., 2014], Nuno-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J., 2013], Damon and Pike [Geom. Topol. 2014].

Milnor number of determinantal surface in $\mathbb{C}^N$,

The Milnor number of X at 0, denoted by $\mu(X)$, is defined as $\mu(X) = b_2(X_u)$, where X_u is the generic fiber of X and $b_2(X_t)$ is the 2 -th Betti number.

Le-Greuel type formula

Proposition: [Math. Scand. 2014], [Israel J. 2013], [Geom. Top. 2014]

Let $(X,0) \subset (\mathbb{C}^N,0)$ be 2-dimensional IDS admitting smoothing. Let $p:(\mathbb{C}^N,0) \to (\mathbb{C},0)$ be a generic linear function on X. Then,

$$\mu(X) + \mu(X \cap p^{-1}(0)) = m_2(X),$$

where $m_2(X)$ is the second polar multiplicity of X.

M.S. Pereira'conjecture, Ph. Thesis (2010)

([Math. Scand. 2014], [Geom. Top. 2014])
If $X^2 \subset \mathbb{C}^4$ is a simple 2-dimensional IDS, then $\mu(X) + 1 = \tau(X)$

Question

Does this formula hold for all 2-dimensional IDS ?

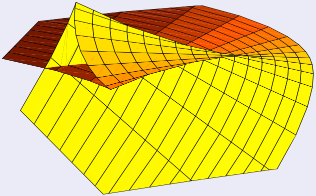
Sections of Determinantal Varieties

Definition

The hyperplane $H \subset \mathbb{C}^N$, given by the kernel of the linear function $p : \mathbb{C}^N \to \mathbb{C}$ is called general with respect to X at 0 if H is not the limit of tangent hyperplanes to X at 0.

General and strongly general hyperplane

Example: Swallowtail



General and strongly general hyperplane

Definition

Let $X \subset \mathbb{C}^N$ be a d- dimensional analytic complex variety, and let $\{V_i\}$ be a stratification of X. The hyperplane $H \subset \mathbb{C}^N$ is called strongly general at the origin if H is general and there exists a neighbourhood U of 0 such that for all strata V_i of X, with $0 \in \overline{V}_i$, we have that $H \pitchfork V_i$ at x, $\forall\, x \in U \backslash \{0\}$.

Proposition

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a d- dimensional EIDS of type (m, n, t). If $H \subset \mathbb{C}^N$ is a strongly general hyperplane then $X \cap H \subset \mathbb{C}^{N-1}$ is a $d-1$-dimensional EIDS of the same type.

Minimality of Milnor number

Let X be a d- dimensional complex variety. A hyperplane H is general if and only if $\mu(X \cap H)$ is minimum.

1. B. Teissier [Astérisque, 7 et 8 , (1973)], Henry and Le Dung Trang, [LNM, 482 (1975)], the case of hypersurfaces.
2. T. Gaffney [Travaux en Cours, **55** (1997)], ICIS.
3. J. Snoussi [Comment. Math. Helv. **76** (1), (2001)], normal surfaces in $\mathbb{C}^N$.

Minimality of Milnor number

Similar result holds for 3-dimensional EIDS.

Theorem

(Chachapoyas-Siésquen)*Let* $X \subset \mathbb{C}^N$ *be a* 3*- dimensional determinantal variety with isolated singularity and let* H *be a hyperplane in* $\mathbb{C}^N$*. Suppose that* $X \cap H$ *has an isolated singular point, then the following conditions are equivalent.*

- H *is general to* X *at* 0*.*
- $\mu(X \cap H)$ *is minimum and* $\mu(X \cap H \cap H')$ *is minimum for all hyperplane* H' *general to* X *and to* $X \cap H$*.*

Sections of EIDS

The following result is a generalization of a result of Lê Dung Trang [Singularity theory, World Sci. Publ., Hackensack, NJ, 2007.] We use the Lê-Greuel formula for surfaces.

Theorem

Let $X \subset \mathbb{C}^N$ be a d- dimensional EIDS. Let H, H' be hyperplanes in $\mathbb{C}^N$ strongly general to $(X, 0)$ at the origin. Then there exist P and P', $P \subset H$ and $P' \subset H'$ such that $\operatorname{codim} P = \operatorname{codim} P' = d - 2$, and the determinantal surfaces $X \cap P$ and $X \cap P'$ satisfy the following conditions:

- **a)** *$X \cap P$ and $X \cap P'$ have isolated singularity.*
- **b)** *$X \cap P$ and $X \cap P'$ admit smoothing.*
- **c)** $\mu(X \cap P) = \mu(X \cap P')$.

Euler obstruction

Theorem: (Brasselet, D. T. Lê, and J. Seade,[Topology 39, (2000)])

Let $(X,0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^N$. Let $\{V_i\}$ be a Whitney stratification of a small representative X of $(X,0)$. Then for a generic complex linear form $l : \mathbb{C}^n \to \mathbb{C}$, and for ϵ and $r \neq 0$ sufficiently small, the following formula for the Euler obstruction of $(X,0)$ holds,

$$Eu_0(X) = \sum_i \chi(V_i \cap B_\epsilon \cap l^{-1}(r)) Eu_{V_i}(X),$$

where the sum is over strata V_i such that $0 \in \overline{V}_i$ and $Eu_{V_i}(X)$ is the Euler obstruction of X in any point of V_i.

Euler obstruction of an EIDS: $N \leq (m-t+3)(n-t+3)$

Let $X = F^{-1}(M^t_{m,n})$ be an EIDS, defined by $F : \mathbb{C}^N \to M_{m,n}$. If $N \leq (m-t+3)(n-t+3)$ then the singular part $\Sigma X = F^{-1}(M^{t-1}_{m,n})$ is an IDS . Then the variety X admits 3 strata $\{V_0, V_1, V_2\}$, $V_0 = \{0\}$, $V_1 = \Sigma X \backslash \{0\}$, $V_2 = X_{reg}$.
Using the previous Theorem, we have

$$Eu_0(X) = \chi(\Sigma X \cap l^{-1}(r) \cap B_\epsilon)(\chi(L_{V_1}) - 1) + \chi(X \cap l^{-1}(r) \cap B_\epsilon).$$

This formula can be expressed in terms of the singular vanishing Euler characteristic.

$$Eu_0(X) = (\tilde{\chi}(\Sigma X \cap l^{-1}(0) \cap B_\epsilon) + 1)(\chi(L_{V_1}) - 1) + \tilde{\chi}(X \cap l^{-1}(0) \cap B_\epsilon) + 1.$$

Euler obstruction

Proposition

Let $X = F^{-1}(M^t_{m,n})$ be an EIDS defined by $F : \mathbb{C}^N \to M_{m,n}$, if $N \leq (n-t+3)(m-t+3)$ and ΣX is an ICIS. Then

$$
\begin{aligned}
Eu_0(X) &= ((-1)^{\dim(\Sigma X \cap l^{-1}(0))}\mu(\Sigma X \cap l^{-1}(0)) + 1)(\chi(L_{V_1}) - 1) \\
&\quad + \tilde{\chi}(X \cap l^{-1}(0) \cap B_\epsilon) + 1
\end{aligned}
$$

where $l : \mathbb{C}^N \to \mathbb{C}$ is a generic linear projection centered at 0, L_{V_1} is the complex link of the stratum V_1 in X and B_ϵ is the ball of radius ϵ in $\mathbb{C}^N$.

Euler obstruction, $F : \mathbb{C}^N \to M_{2,3}$

Theorem

Let $X = F^{-1}(M^2_{2,3}) \subset \mathbb{C}^N$ be an EIDS defined by the function $F : \mathbb{C}^N \to M_{2,3}$ with $N = 6$. Then

$$Eu_0(X) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)) + 1 = \chi(X \cap l^{-1}(r)).$$

Theorem

Let X as above, but with $N \geq 7$. Then

$$Eu_0(X) = (-1)^{N-7}\mu(\Sigma X \cap l^{-1}(0)) + \tilde{\chi}(X \cap l^{-1}(0)) + 2.$$

If F has corank 1, then

$$Eu_0(X) = 2.$$

Happy Birthday, Pepe !!